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## ON A DETERMINANT EACH OF WHOSE ELEMENTS IS THE PRODUCT OF $k$ FACTORS.

By PROF. W. H. METZLER, Syracuse University, Syracuse, N. Y.

1. In Muir's *Theory of Determinants*, page 117, the following example (slightly modified) is given without comment :

$$\begin{vmatrix} h_1 m_1 & h_1 n_1 & k_1 x_1 & k_1 z_1 \\ h_2 m_1 & h_2 n_1 & k_2 x_1 & k_2 z_1 \\ p_1 m_2 & p_1 n_2 & q_1 x_2 & q_1 z_2 \\ p_2 m_2 & p_2 n_2 & q_2 x_2 & q_2 z_2 \end{vmatrix} = \begin{vmatrix} h_1 & k_1 & 0 & 0 \\ h_2 & k_2 & 0 & 0 \\ 0 & 0 & p_1 & q_1 \\ 0 & 0 & p_2 & q_2 \end{vmatrix} \cdot \begin{vmatrix} m_1 & 0 & m_2 & 0 \\ n_1 & 0 & n_2 & 0 \\ 0 & x_1 & 0 & x_2 \\ 0 & z_1 & 0 & z_2 \end{vmatrix}$$

$$= \begin{vmatrix} h_1 & k_1 \\ h_2 & k_2 \end{vmatrix} \cdot \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \cdot \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix} \cdot \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}.$$

From this it is seen that if we take  $n_1$  determinants

$$\left| b_{1n_1}^{(n_1, j_1)} \right|, (j_1=1, 2, \dots, n_1),$$

of order  $n_2$  and form a determinant of order  $n=n_1 n_2$  by arranging them along the principal diagonal as in the first factor of the right-hand member of the first equation in the above example, all the other elements being zeros, and  $n_2$  determinants

$$\left| b_{1n_1}^{(n_1, j_2)} \right|, (j_2=1, 2, \dots, n_2),$$

of order  $n_1$  and form another determinant of order  $n$  in which the elements in the  $(\alpha n_2 + \beta)$ th column of the  $\beta$ th  $n_1$  rows are the elements of the  $(\alpha+1)$ th column of

$$| b_{1n_1}^{(n_1, \beta)} |, (\alpha=0, 1 \dots n-1), (\beta=1, 2 \dots n_2),$$

all the other elements being zeros, the product of these two determinants of the  $n$ th order is a determinant of the  $n$ th order whose elements are products of two factors.

It is also seen that the elements of  $| b_{1n_2}^{(n_1, j_1)} |$  are found in all the columns, but in the  $j_1$ th  $n_2$  rows only of the product, and that the elements of  $| b_{1n_1}^{(n_1, j_2)} |$  are found in all the rows, but in the  $(\alpha n_2 + j_2)$ th columns only of the product.

2. We may now form two determinants of the  $n$ th order ( $n=n_1 n_2 n_3$ ) precisely as in Art. 1, first by taking  $n_3$  determinants of order  $n_1 n_2$  each of whose elements is the product of two factors as there found, and second by taking  $n_1 n_2$  determinants

$$| b_{1n_3}^{(n_3, j_3)} |, (j_3=1, 2 \dots (n/n_3))$$

of order  $n_3$ , and the product of these two determinants of the  $n$ th order will be a determinant of the  $n$ th order each of whose elements is the product of three factors, one factor an element from a determinant of each of the three orders.

Continuing this process we arrive at a determinant of the order  $n=n_1 n_2 \dots n_k$ , each of whose constituents is the product of  $k$  factors, one factor an element from a determinant of each of the  $k$  given orders  $n_1, n_2 \dots n_k$

3. Let  $\left\{ \frac{\alpha}{\beta} \right\}$  denote the greatest integer in  $\frac{\alpha}{\beta}$ ,  $R \frac{\alpha}{\beta}$  denote the remainder on dividing  $\alpha$  by  $\beta$ ,

$$| b_{1n_g}^{(n_g, j_g)} |, (g=1, 2 \dots k), (j_g=1, 2 \dots n/n_g),$$

denote the  $n/n_g$  determinants of order  $n_g$ . Then if  $n=n_1 n_2 \dots n_k$  we have the theorem

$$A = | a_{1n} | = II | b_{1n_g}^{(n_g, j_g)} |, (g=1, 2 \dots k), (j_g=1, 2 \dots n/n_g) \dots (1).$$

Where the element in the  $x$ th row and  $y$ th column of  $A$ ,

$$a_{xy} = b \left( n_1, \left\{ \frac{x-1}{n_1} \right\} + 1 \right) R \frac{x-1}{n_1} + 1, \left\{ \frac{y-1}{n/n_1} \right\} + 1$$

$$\prod_{h=2}^{h=k} b \left( n_h, \left\{ \frac{x-1}{n_1 n_2 \dots n_h} \right\} n_1 n_2 \dots n_{h-1} + \left\{ \frac{y-1}{n} \right\} + 1 \right) \dots (2).$$

$$\left\{ \frac{R \frac{y-1}{n}}{n_1 n_2 \dots n_{h-1}} \right\} + 1, \left\{ \frac{R \frac{x-1}{n_1 n_2 \dots n_h}}{n_1 n_2 \dots n_{h-1}} \right\} + 1$$

$$\left( \frac{n}{n_1 n_2 \dots n_h} \right)$$

Each of the elements  $b_{\alpha_g \beta_g}^{(n_g, i_g)}$  ( $\alpha_g, \beta_g = 1, 2 \dots n_g$ ) occurs in  $\frac{n}{n_g \dots n_k}$  rows and in  $\frac{n}{n_1 \dots n_g}$  columns of  $A$ .

If we write  $i_g = \lambda_g + \mu_g$ , where

$$\left\{ \lambda_g = 0, 1 \dots \left( \frac{n}{n_1 \dots n_g} - 1 \right) \right\},$$

$$\left\{ \mu_g = 1, 2 \dots (n_1, n_2 \dots n_{g-1}) \right\},$$

and understand that when  $g=1$ ,  $n_1 n_2 \dots n_{g-1} = 1$ .

Then the element  $b_{\alpha_g \beta_g}^{(n_g, \lambda_g n_1 \dots n_{g-1} + \mu_g)}$  occurs in the  $(\lambda_g n_g + \alpha_g)$ th  $n_1 n_2 \dots n_{g-1}$  rows, and in the  $\{(\mu_g - 1)n_g + \beta_g\}$ th  $\frac{n}{n_1 \dots n_g}$  columns of  $A$ .

4. If all the determinants of the same order  $n_g$ , ( $g=1, 2 \dots k$ ) are equal the theorem becomes

$$A = \prod | b_{1 n_g}^{(n_g)} |^{i_g}, i_g = n/n_g \dots (3).$$

If  $n_1 = n_2 = n_3 = \dots = n_k = m$ , then  $n = m^k$  and the theorem takes the form

$$A = \prod | b_{1 m}^{(h)} |, (h=1, 2 \dots k, m^{k-1}) \dots (4).$$

If all the determinants are of the same order and equal to each other then the theorem becomes

$$A = | b_{1 m} |^{k \cdot m^{k-1}} \dots (5).$$

After finding the general theorem in Art. 3, the special cases (3) and (5) were first made known to me by Prof. E. H. Moore, who discovered them before knowing of the general theorem, and who immediately made the generalization, using the different notation, on receiving the Muir reference.

December 5, 1898.